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Conditional Stability of Multiple-Charged Solitons

Yu. P. Rybakov and S. Chakrabarti

Department of Theoretical Physics, Patrice Lumumba University, GSP 117923, Moscow V-302, 3, Ordjonikidze Street, USSR

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The stability of the three-dimensional multiple-charged soliton solutions to the nonlinear field equations is studied by Lyapunov's method. It is proved that an absolutely stable soliton solution can not exist in any field model. By imposing the subsidiary condition $\omega^l \delta Q_l = 0$ (fixation of charges) we find a sufficient condition for stability of the stationary soliton which includes the inequality $\omega^k \omega^i (\partial Q^i / \partial \omega^k) < 0$. An illustrative example is considered.

1. GENERAL STABILITY CONDITIONS FOR THREE-DIMENSIONAL SOLITONS

In the last ten years the soliton solutions to the nonlinear field equations have found numerous applications in various branches of physics. In particular, one can notice the increasing interest in Einstein's geometrical program, which included the use of soliton solutions for the description of extended particles (Faddeev, 1979). In this paper we mean by soliton the localized regular solution to field equations with finite energy and other dynamic characteristics.

One of the important problems in soliton physics is the study of soliton stability. Here we shall study the stability of realistic three-dimensional solitons in the Lyapunov sense, i.e., stability with respect to shape (Benjamin, 1972; Rybakov and Chakrabarti, 1981).

Let $\varphi = \{\varphi^s(t, \bar{r})\}$, $s = 1, \ldots, n$, be a real n-component field satisfying the second-order equations of motion. Let $\varphi = u$ be stationary (periodic in time) soliton solution, which is supposed to be localized in the domain of

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unity size. This signifies that u has the following asymptotic behavior at $r = |\bar{r}| \to \infty$:

$$u(t, \bar{r}) = O(e^{-r}) \tag{1}$$

Let U be the set of fields obtained from u by 3-translations, 3-rotations, and admitted gauge transformations. The soliton solution will be called perturbed with respect to shape only if $\varphi \notin U$. It is clear that in this case φ will be nonstationary solution to the field equations. Let us denote the perturbation of the soliton solution u by $\xi = \varphi - u$. Following Rybakov and Chakrabarti (1981) we introduce two functional metrics: $\rho_0[\xi_0(\bar{r})]$ and $\rho[\xi(t, \bar{r})]$ for the characterization of the initial and current perturbations, respectively. We assume that the metric ρ is continuous with respect to the metric ρ_0 and $\rho(\xi) = 0$ if $\varphi \in U$.

Soliton solution u is called stable in the Lyapunov sense with respect to the metrics ρ_0 and ρ, if for any $\varepsilon > 0$, one can find $\delta(\varepsilon) > 0$ such that for $\rho_0(\xi_0) < \delta$ the inequality $\rho(\xi) < \varepsilon$ holds for any $t > 0$. The study of stability of stationary solitons in scalar (Rybakov, 1979) and spinor (Rybakov, 1965) models showed that the absolutely stable stationary solitons cannot exist.

Therefore it is reasonable to consider the conditional stability of solitons, that is, to impose some subsidiary conditions on the initial perturbations ξ_0. This procedure seems to be inevitable to provide the stability of solitons. To demonstrate the general character of this fact, we show in this paper that in any realistic three-dimensional model only conditionally stable solitons can exist.

> *Theorem 1.* In translationally invariant theory the stationary solitons cannot be absolutely stable in the Lyapunov sense (Rybakov, 1983).

Proof. It can be noted (Movchan, 1960) that for stability of soliton solution u with respect to the metrics ρ_0, ρ it is necessary and sufficient that there exist a Lyapunov functional $V[\varphi]$ with the following properties:

1. The soliton solution u is a stationary point of V.
2. V is definitely positive with respect to the metric ρ in some neighborhood of u.
3. V is continuous with respect to the metric ρ_0.
4. V does not increase along the trajectories of motion.

Assuming the stability of the solution u, we shall show that the additive functional V with the properties mentioned above does not exist.

Let us take into account that in a translationally invariant theory the additive Lyapunov functional should have the form

$$V[\varphi] = \int d^3x U(\dot{\varphi}, \partial_i \varphi, \varphi), \qquad i = 1,2,3 \tag{2}$$

where $\dot{\varphi} = \partial \varphi / \partial t$.

We introduce the following notifications:

$$U_s^{\cdot} = \partial U / \partial \dot{\varphi}^s, \qquad U_s^i = \partial U / \partial (\partial_i \varphi^s), \qquad U_s = \partial U / \partial \varphi^s$$

The field u being the stationary point of the functional V, it satisfies the equations

$$U_s^{\cdot} = 0, \qquad U_s - \partial_i U_s^i = 0 \tag{3}$$

Write down the second variation of the functional V in the neighborhood of u:

$$\delta^2 V[\dot{\xi}, \xi] = \int d^3x \{ U_{sr}^{\cdot\cdot} \dot{\xi}^s \dot{\xi}^r + 2U_{sr}^{\cdot} \dot{\xi}^s \xi^r + U_{rs} \xi^s \xi^r$$

$$+ 2U_{sr}^{\cdot i} \dot{\xi}^s \partial_i \xi^r + U_{rs}^{ik} \partial_i \xi^r \partial_k \xi^s + 2U_{rs}^i \partial_i \xi^r \xi^s \} \tag{4}$$

Let us put in (4) the special perturbations:

$$\dot{\xi}^s = f^j \partial_j \dot{u}^s, \qquad \xi^s = f^j \partial_j u^s$$

where $f^j(\bar{r})$ represents sectionally smooth functions. Using the equations (3), $\delta^2 V$ can be transformed to

$$\delta^2 V[\bar{f}] = \int d^3x [\partial_i f^l A_{lj}^{ik} \partial_k f^j + (\partial_i f^l f^j - f^l \partial_i f^j) B_{jl}^i] \tag{5}$$

where

$$A_{lj}^{ik} = \partial_l u^r U_{rs}^{ik} \partial_j u^s, \qquad 2B_{jl}^i = \partial_{[j} U_r^i \partial_{l]} u^r = -2B_{lj}^i \tag{6}$$

Note that for the positive definiteness of $\delta^2 V$ it is necessary that in (4) the quadratic form in derivatives $U_{sr}^{ik} \partial_i \xi^r \partial_k \xi^s$ would be positive (the Legendre–Hadamard condition). Hence the first term in (5) is positive, while the second one is evidently sign changing. Using the equation (3), it is not difficult to obtain the relation

$$\partial_i B^i_{jl} = 0$$

from which it follows that

$$2B^i_{jl} = \varepsilon^{ikn} \partial_k a_{njl} \tag{7}$$

where

$$a_{njl} = \frac{\varepsilon_{nki}}{2\pi} \partial^k \int d^3x' \frac{B^i_{jl}(\bar{r}')}{|\bar{r} - \bar{r}|'} \tag{8}$$

Performing in (5) the integration by parts and using (7) we get

$$\delta^2 V[\bar{f}] = \int d^3x \left[\partial_i f^l \left(A^{ik}_{lj} + \varepsilon^{ikn} a_{njl} \right) \partial_k f^j \right] \tag{9}$$

Now it is sufficient to ascertain the sign-changing character of the integrand in (9). With this aim let us consider the asymptotic region $r \to \infty$. In this case from (8) it follows that $a_{njl} = O(r^{-3})$, while from (1) and (6) we get that $A^{ik}_{lj} = O(e^{-2r})$. Thus we conclude that the integrand in (9) is sign changing for $r \to \infty$, which contradicts the initial assumption on stability of the soliton solution u. The theorem is proved.

In light of this result one can only speak of conditional stability of stationary solitons.

2. CONDITIONAL STABILITY OF MULTIPLE-CHARGED SOLITONS

Let us search for the sufficient criteria of conditional stability of stationary solitons. Let us consider the real n-component field $\Psi = \{\Psi^s\}$, $s = \overline{1, N}$, and the corresponding Lagrangian density

$$L = -F(\dot{\Psi}, \partial_i \Psi, \Psi)$$

assumed to be invariant under the orthogonal transformations

$$\Psi \to \exp(\alpha^l \Gamma_l) \Psi, \qquad \Gamma_l^T = -\Gamma_l, \qquad l = \overline{1, n}, \tag{10}$$

with the commuting generators Γ_l:

$$[\Gamma_l, \Gamma_k]_- = 0$$

The transformations (10) imply the existence of the conserving charges Q_l, the fixation of which can be considered as the natural subsidiary conditions on the initial field perturbations.

The nonperturbed soliton solution will be assumed of the form

$$\Psi_{(0)} = \exp(\hat{\omega} t) u \tag{11}$$

where $\hat{\omega} = \omega^l \Gamma_l$, ω^l being constant frequency parameters. In view of (11) it is convenient to take the field function Ψ of the perturbed soliton in the form

$$\Psi = \exp(\hat{\omega} t) \varphi$$

which reduces the Lagrangian density to

$$L = -F(\hat{\omega}\varphi + \dot{\varphi}, \partial_i \varphi, \varphi)$$

Let us choose the Lyapunov functional V as the linear combination of the energy E and the charges Q_l:

$$V = E - \omega^l Q_l = \int d^3x (F - F_s^{\cdot} \dot{\varphi}^s) \tag{12}$$

where

$$Q_l = -\int d^3x [F_s^{\cdot} (\Gamma_l \varphi)] \tag{13}$$

It can be easily seen that the state (11) is a stationary point of the functional (12), satisfying the equations

$$\partial_i F_s^i - F_s - F_r^{\cdot} \hat{\omega}_s^r = 0 \tag{14}$$

Let us impose the following subsidiary condition on the perturbations $\xi^s = \varphi^s - u^s$:

$$\omega^l \delta Q_l = 0 \tag{15}$$

i.e., the fixation of the quantity $\Sigma_1^n \omega^l Q_l$. Write down the second variation of the functional (12) in the neighborhood of u:

$$\begin{aligned} \delta^2 V = \int d^3x \{ & -F_{rs}^{\cdot\cdot} \dot{\xi}^s \dot{\xi}^r + F_{sr}^{\cdot\cdot} (\hat{\omega}\xi)^s (\hat{\omega}\xi)^r \\ & + 2F_{sr}^{\cdot i} (\hat{\omega}\xi)^s (\partial_i \xi^r) + 2F_{sr}^{\cdot} (\hat{\omega}\xi)^s \xi^r \\ & + F_{rs} \xi^r \xi^s + F_{sr}^{ik} \partial_i \xi^s \partial_k \xi^r + 2F_{sr}^i \partial_i \xi^s \xi^r \} \end{aligned} \tag{16}$$

and the condition (15)

$$-\int d^3x(F_{sr}^{\cdot\cdot}\dot{\xi}^r)=(g,\xi)$$

$$=\int d^3x\{[F_{sr}^{\cdot\cdot}(\hat{\omega}\xi)^r+F_{sr}^{\cdot k}\partial_k\xi^r+F_{sr}^{\cdot}](\hat{\omega}u)^s+F_s^{\cdot}(\hat{\omega}\xi)^s\} \tag{17}$$

As is clear from (16), for definite positiveness of δ^2V the positiveness of the form $-F_{sr}^{\cdot\cdot}\dot{\xi}^s\dot{\xi}^r$ is necessary. Hence there exists the square root A of the matrix $-F_{sr}^{\cdot\cdot}=(A^2)_{sr}$. Using the Schwartz inequality

$$(y,y)(Ax,Ax)\geqslant(Ax,y)^2$$

for $x^s=\dot{\xi}^s$, $y^s=(A\hat{\omega}u)^s$ and representing δ^2V in the following way,

$$\delta^2V=(\dot{\xi},A^2\dot{\xi})+(\xi,\hat{L}\xi) \tag{16$'$}$$

we get the estimate

$$\delta^2V\geqslant(\xi,\hat{L}\xi)+a^{-1}(g,\xi)^2\equiv(\xi,\hat{K}\xi) \tag{18}$$

Here the self-conjugate operators $\hat{K}$ and $\hat{L}$ are introduced and

$$a=-\int d^3x\{F_{sr}^{\cdot\cdot}(\hat{\omega}u)^s(\hat{\omega}u)^r\}>0$$

Taking the special perturbation

$$\xi^s=\omega^l(\partial u^s/\partial\omega^l)\equiv v^s \tag{19}$$

and using the equations of motion (14) differentiated by ω^l, the following relation can be deduced:

$$(\hat{K}v)^s=-\frac{1}{a}\omega^l\omega^m(\partial Q_m/\partial\omega^l)\{-\partial_k[F_{rs}^{\cdot k}(\hat{\omega}u)^r]$$

$$+F_r^{\cdot}\hat{\omega}_s^r+(\hat{\omega}u)^r(F_{rt}^{\cdot\cdot}\hat{\omega}_s^t+F_{rs}^{\cdot})\} \tag{20}$$

Introducing the designation

$$f(\omega)=\omega^l\omega^m(\partial Q_m/\partial\omega^l)$$

we find from (20) and (13) that

$$(v, \hat{K}v) = (f/a)(f - a) \tag{21}$$

On the other hand, if $\lambda(\bar{\omega})$ is the first eigenvalue of the operator $\hat{K}$, then from (18) it follows that the stability domain Ω_0 in $\bar{\omega}$ space is defined by the inequality $\lambda(\bar{\omega}) > 0$. Let us assume that in some region $\Omega \supset \Omega_0$, $\lambda(\bar{\omega})$ is the single eigenvalue, which can be negative. If in Ω, the inequality $\omega^l(\partial\lambda/\partial\omega^l) \geqslant 0$ holds, then from (20) it follows that the boundary Σ_0 of the stability domain is defined by the equality $f(\bar{\omega}) = 0$ and $v^s|_{\Sigma_0}$ is the eigenfunction of the operator $\hat{K}$ corresponding to the eigenvalue $\lambda = 0$. Now it is seen from (21) that the stability domain Ω_0 is defined by the inequality

$$f(\bar{\omega}) = \omega^l\omega^m(\partial Q_m/\partial\omega^l) < 0 \tag{22}$$

Thus we have proved the following statement generalizing for the multiple-charged solitons the well-known theorem on Q stability of scalar stationary solitons (Kumar et al., 1979; Makhankov, 1978).

> *Theorem 2.* If there exists a domain Ω in $\bar{\omega}$ space such that (1) the operator $\hat{K}$ has a single eigenvalue $\lambda(\bar{\omega})$ which can be negative, (2) $\lambda(\bar{\omega})$ increases along the $\bar{\omega}$ direction, then the stability domain $\Omega_0 \subset \Omega$ can be defined by the inequality (22).

3. ILLUSTRATIVE EXAMPLE

Let us demonstrate an example illustrating the application of the Theorems 1 and 2. Consider the isospinor Synge's model defined by the Lagrangian density

$$L = \tfrac{1}{2}\partial_\mu\varphi^+\partial^\mu\varphi - \tfrac{1}{2}\varphi^+\varphi + \tfrac{1}{3}(\varphi^+\varphi)^{3/2}$$

where $\varphi^T = (\varphi_1, \varphi_2)$, the isospinor field. The equations of motion admit the stationary solution of the form

$$\varphi_k^{(0)} = u_k(r)\exp(-i\omega_k t), \qquad u_k^* = u_k \tag{23}$$

where u_k satisfies the equation

$$(\Delta - 1 + \omega_k^2 + u)u_k = 0, \qquad u = (u_1^2 + u_2^2)^{1/2}, \qquad k = 1,2 \tag{24}$$

From (24) one can easily derive the relation

$$(\omega_1^2 - \omega_2^2)(u_1, u_2) = 0$$

which indicates that for $\omega_1 \neq \omega_2$ one of the functions u_k, must have a node. In particular, if $\omega_1^2 > \omega_2^2$ then by Courant's theorem (Courant and Hilbert, 1953) $u_2 > 0$ and u_1 is nodal. Defining the perturbation of the soliton solution by

$$\xi_k + i\eta_k = \varphi_k \exp(i\omega_k t) - u_k$$

and taking the special perturbation $\xi_1 \neq 0$, $\eta_1 \neq 0$, $\xi_2 = \eta_2 = 0$, we come to the scalar case of perturbed nodal stationary solution $\varphi_1^{(0)} = u_1 \exp(-i\omega_1 t)$. As was shown in the paper by Kumar et al. (1979) such a solution is Q unstable. Therefore we conclude that only for $\omega_1 = \omega_2 = \omega$ the stationary soliton may be stable. In this case the Lyapunov functional takes the form

$$\begin{aligned} V &= E - \omega_1 Q_1 - \omega_2 Q_2 \\ &= E - \omega(Q_1 + Q_2) \end{aligned} \tag{25}$$

and the solutions to the equations (24) can be represented as $u_1 = u\cos\mu$; $u_2 = u\sin\mu$, $\mu = \text{const}$. Taking into account the results of Section 2, we can obtain the following estimate for the second variation of the functional (25):

$$\delta^2 V \geqslant \sum_{i=1}^{4} (h_i, \hat{K}_i h_i)$$

where the perturbations are redefined:

$$h_1 = \xi_1 \cos\mu + \xi_2 \sin\mu, \qquad h_2 = -\xi_1 \sin\mu + \xi_2 \cos\mu$$

$$h_3 = \eta_1, \qquad h_4 = \eta_2$$

and the following self-conjugated operators are introduced:

$$\hat{K}_1 = \hat{K}_2 - u + 4\omega^2 P_u, \qquad \hat{K}_2 = \hat{K}_3 = \hat{K}_4 = 1 - \omega^2 - \Delta - u \tag{26}$$

Here P_u denotes the projector on u.

The spectrum of the operators $\hat{K}_i$ was investigated in the paper by Kumar et al. (1979), where it was shown that the domain of Q stability is defined by the inequality $|\omega| > 1/\sqrt{2}$ in accordance with the Theorem 2.

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